COMBINATORICA
Bolyai Society – Springer-Verlag

Combinatorica **25** (1) (2005) 85–103

CLASSIFICATION OF LOCALLY 2-CONNECTED COMPACT METRIC SPACES

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Received August 31, 2001

The aim of this paper is to prove that, for compact metric spaces which do not contain infinite complete graphs, the (strong) property of being "locally 2-dimensional" is guaranteed just by a (weak) local connectivity condition. Specifically, we prove that a locally 2-connected, compact metric space M either contains an infinite complete graph or is surface like in the following sense: There exists a unique surface S such that S and M contain the same finite graphs. Moreover, M is embeddable in S, that is, M is homeomorphic to a subset of S.

1. Introduction

The classification of surfaces is a corner-stone in topology. A *surface* is a compact Hausdorff space which is locally homeomorphic to the Euclidean plane. The classification theorem says that every surface is homeomorphic to either S_g, the sphere with g handles added, or to N_k, the sphere with k crosscaps added. A short proof was given in [5], see also [2]. The purpose of the present paper is to extend this classification to a much more general class of spaces, namely those compact metric spaces in which the rather strong property of being locally 2-dimensional are replaced by two apparently much weaker conditions: We are going to assume that

(i): the metric space M under consideration does not contain an infinite complete graph, and

Mathematics Subject Classification (2000): 05C10, 57M15

0209–9683/105/$6.00

(ii): M is *locally* 2*-connected* which means that, for every element x in the space, and every neighborhood U around x, there is a neighborhood V of x contained in U such that both V and $V \setminus x$ are connected.

The mild condition (i) is to exclude M from being e.g. a compact subset of a higher dimensional space. The property of being locally connected is often imposed on a topological space. It has the convenient implication that "connected" is the same as "arcwise connected", see e.g. [1], page 116. (Note that the neighborhood V in (ii) can be chosen to be open and hence arcwise connected. Then also $V \setminus x$ is open and hence arcwise connected.)

The main point of the present work is that the condition (ii), together with (i) imply that the space is locally 2-dimensional in the sense that M is embeddable in a surface S_g or N_k. We say that the *Euler genus* of S_g and N_k, respectively, are $2g$ and k, respectively. Many complete graphs are embeddable in two different surfaces of the same smallest possible Euler genus, see [2], and therefore also in all surfaces of larger Euler genus. However, if we embed the space M satisfying (i) and (ii), in a surface S of smallest possible Euler genus, then M and S contain the same graphs. As no two nonhomeomorphic surfaces contain precisely the same graphs, see e.g. [2], it follows that S is unique. Moreover, it also follows from the classification theorem that either $M = S$ or else M can be embedded into precisely those surfaces which are obtained from S by adding handles or crosscaps.

The notation and terminology are the same as in [6]. In particular, we call a simple arc an SA and a simple closed curve an SCC. If C is an SCC in the plane or a contractible SCC on a surface, then its interior is denoted $int(C)$ and its exterior $ext(C)$. We put $Int(C) = int(C) \cup C$ and $Ext(C) = ext(C) \cup C$. For the definition of the *complete graph* K_n, a *bipartite graph*, the *bipartite graph* $K_{3,3}$, and *overlap graph*, see [6]. If x is an element in a metric space, and r is a positive real number, then $D(x,r)$ is the open disc with radius r, and $C(x,r)$ is the set of elements of distance r from x. A *surface with holes* is obtained from a surface by deleting the interiors of a possibly infinite collection of pairwise disjoint contractible SCC-s. It follows easily from the classification theorem that, if M is a compact Hausdorff space such that, for each element x of M, there is a neighborhood around x which is homeomorphic to a closed disc (where x may be on the boundary, in which case we may also think of the neighborhood as a closed half-disc) in the plane, then M is homeomorphic to a surface with a finite number of holes.

We say that a topological space M_1 contains a topological space M_2 if M_2 is homeomorph to a subspace of M_1. Note that every finite graph can be regarded as a metric space when every edge is regarded as the unit interval on the real line. Thus the space M_1 contains a finite graph G if the vertices of G can be represented by distinct points in M_1 and edges in G are SA-s which are pairwise disjoint except for their ends. We also use this definition when we say that M_1 contains an infinite complete graph. (In other words, we do not consider the infinite complete graph to be a topological space.)

2. Properties of locally 2-connected, compact metric spaces

We first observe that local 2-connectivity is a rather strong connectivity property in that it implies infinite connectivity in the following sense:

Proposition 2.1. *If M is a connected, locally 2-connected metric space, and Q is a finite subset of M, then $M \setminus Q$ is arcwise connected.*

Proof. Let x, y be in $M \setminus Q$. Let A be an SA in M connecting x and y. If z is in $A \cap Q$, then we select a neighborhood U_z around z such that $U \setminus \{z\}$ is connected and disjoint from Q. We can now successively reroute A in each such U_z and obtain an SA from x to y in $M \setminus Q$. ∎

Proposition 2.2. *Let S be a surface, and let $\mathcal{C}$ be a collection of contractible simple closed curves in S such that no element of S is in the interior of two distinct curves in $\mathcal{C}$. Let (M, d) be the compact space obtained from S by deleting the interior of each curve in $\mathcal{C}$. Then M is locally 2-connected if and only if all the curves in $\mathcal{C}$ are pairwise disjoint and, for each positive number ϵ, only finitely many curves of $\mathcal{C}$ have diameter $> \epsilon$. If M is locally 2-connected, then M and S contain the same finite graphs.*

Proof. We first prove the "only if" part. So we assume that M is locally 2-connected.

If C is a curve in $\mathcal{C}$, then S has another contractible simple closed curve C' having C in the interior. Using the Jordan–Schönflies theorem, see e.g. [5], we may think of C, C' as concentric circles in the Euclidean plane. Now, if $\mathcal{C}$ has another curve intersecting C in x, say, then clearly M is not locally 2-connected at x. So the curves in $\mathcal{C}$ are pairwise disjoint.

Now consider any positive number ϵ and suppose (reductio ad absurdum) that infinitely many curves in $\mathcal{C}$ have diameter $> \epsilon$. Pick an element on each curve of diameter $> \epsilon$ in $\mathcal{C}$. Since we picked infinitely many elements, they have an accumulation point x. Let U be a neighborhood of diameter $< \epsilon$ around x. We may think of U as a closed disc in the plane of radius 3. For each natural number n, there is a curve C_n in $\mathcal{C}$ containing an SA C'_n in U which connects two points on the boundary of U and which contains a point of distance at most $1/n$ from x. The circles of radius 1 and 2, respectively, contain an accumulation point y and z, respectively, of the arcs C'_n. If M is locally connected at x, y, z, then we can find a drawing of $K_{3,3}$ in U as in the proof of statement (2) in [6]. This contradiction proves that only finitely many curves in $\mathcal{C}$ have diameter $> \epsilon$.

This proves the "only if" part of Proposition 2.2.

We now assume that all the curves in $\mathcal{C}$ are pairwise disjoint and, for each positive number ϵ, only finitely many curves of $\mathcal{C}$ have diameter $> \epsilon$. We first prove that M and S contain the same (finite) graphs. Let G be any graph on S. We shall modify the embedding of G in S to an embedding of G in M. First we observe that M has uncountably many elements which are not contained in any curve in $\mathcal{C}$. In fact, if R is any SA in S (which we may think of as a line segment in the plane) which intersects at least two curves in $\mathcal{C}$, then $M \cap R$ has uncountably many elements which are not contained in any curve in $\mathcal{C}$. This is proved in the same way as the Cantor set is proved to be uncountable. Thus we redraw G on S such that no vertex of G is on any curve in $\mathcal{C}$. We shall now reroute the edges which intersect the interior of the curves in $\mathcal{C}$. Consider now a curve C of largest diameter in $\mathcal{C}$ whose interior intersects at least one edge of G. If C intersects only one edge e of G, then we follow e from each of its two ends until we hit C, and then we reroute e along C. In this way e may be modified infinitely many times, but the modifications result in an SA since the diameters of both the deleted and added arcs tend to zero. So we may assume that the interior of C intersects at least two edges of G. We say that an arc A in C *is consistent with the edge* e in G if the ends of A are in e, and A intersects no edge of G distinct from e. A *maximal arc consistent with the edge* e is an arc consistent with the edge e and not a proper subset of another arc consistent with the edge e. There are only finitely many maximal arcs in C consistent with an edge e. (For any two disjoint maximal arcs corresponding to e are connected by an arc in C and any such arc intersects another edge, by the maximality. So, if there were infinitely many maximal arcs, then there would be two edges having a common articulation point on C, and such an accumulation point would be a vertex of G on C, which, however, does not exist.) We let each of these maximal arcs replace the corresponding part of e. After this modification the intersection of G and $Int(C)$ consists of only a finite number of SA-s. If we think of C as a circle in the plane, then it is easy to find a maximal arc A in C such that A is part of an edge e in G and e hits C in the ends of A from the exterior of C. We modify e by replacing A by a part A' of a circle of radius r' only slightly larger than the radius r of C. If r' is sufficiently close to r, then A' intersects only curves in $\mathcal{C}$ of small diameter, and these curves intersect no edge of G, except that arbitrarily small curves in $\mathcal{C}$ may intersect e in elements close to the ends of A. So we choose r'

to be sufficiently close to r, we replace A by A', and then we reroute those parts of A' which intersect the interior of curves in $\mathcal{C}$. The modified part of e corresponding to A will never be modified later in the process. We continue this process for C. Since there are now only finitely many arcs of edges of G in $Int(C)$, the process will terminate for C, and we have modified G so that it does not intersect $int(C)$. We then continue this argument for the curve C of largest diameter in $\mathcal{C}$ whose interior intersects at least one edge of the new representation of G. Continuing like this we modify G into a graph in M. For, an edge may be modified infinitely often. However, the diameters of the modified parts get smaller and smaller. And a modified part is never modified at a later stage. Therefore, each edge converges to an SA. (We leave the technical details to the reader.)

It only remains to prove that M is locally 2-connected. Let x be any element in M, and let U be any neighborhood in M around x. Consider first the case where x is not lying on any curve of $\mathcal{C}$. We may think of U as the interior of a circle C in the Euclidean plane with x as center. Let $\mathcal{C}'$ be those curves in $\mathcal{C}$ which intersect C. Let V be the set of elements in U which can be reached from x by an SA not intersecting any curve in $\mathcal{C}'$. Clearly V is an open, connected neighborhood around x in the plane, and by rerouting arcs, we see that $M \cap V$ is an open, connected neighborhood of x in M. (The rerouting in the present argument is easier than in the previous argument where we rerouted edges because we now just reroute along the curves not in $\mathcal{C}'$.) We claim that also $M \cap (V \setminus \{x\})$ is arcwise connected. Let $y, z \in M \cap (V \setminus \{x\})$ and let A be an SA in $V \setminus \{x\}$ connecting y and z. The arc A may intersect infinitely many curves of $\mathcal{C}$. But, each of these is contained in $int(C)$, and therefore we can reroute A along the boundary. This results in an SA in $M \cap (V \setminus \{x\})$ connecting y and z. Hence M is locally 2-connected at x when x is not in any curve of $\mathcal{C}$. If x is in some curve of $\mathcal{C}$ we argue similarly except that now we may think of U as a half-disc in the Euclidean plane where x is the midpoint of the straight line segment corresponding to part of the curve in $\mathcal{C}$ containing x and bounding the half-disc. This completes the proof of Proposition 2.2. ∎

Consider a circle C and its interior in the Euclidean plane. If we delete the interior of a collection of pairwise disjoint circles in $int(C)$, then the resulting space M is a locally 2-connected, compact metric space by Proposition 2.2. Proposition 2.2 implies that M is dense in the sense that it contains all planar graphs. On the other hand, M may be sparse in the sense that it can be chosen such that it has Lebesgue measure zero. Similar constructions are possible on higher surfaces.

We shall use the following version of *Menger's theorem*.

Theorem 2.3. *Let k be a natural number, and let A,B be disjoint subsets of a metric space M, both with k elements. Assume that, for every set E with at most $k-1$ elements, $M\setminus E$ has an SA from A to B. Then M has k pairwise disjoint $SA-s$ from A to B.*

Proof. Menger's theorem is included in most text books on graph theory, and many proofs apply to the present case. For the sake of completeness, we indicate a proof, by induction on k. We may assume that $k>1$ and that M has $k-1$ pairwise disjoint $SA-s$ $P_1,P_2,\ldots,P_{k-1}$ from A to B. By shortening some of them, if necessary, we may assume that only their ends are in $A\cup B$. So, there is an element a in A and an element b in B not contained in any of $P_1,P_2,\ldots,P_{k-1}$. An *augmenting arc* P from a is a curve P starting at a such that $P\cap(P_1\cup P_2\cup\ldots\cup P_{k-1})$ is a finite number of $SA-s$ none of which is a single point. Moreover, the forward direction of any of these on P is the backward direction on the P_i that it is contained in. In other words, if we follow P from a, and if we hit a P_i, then we proceed backwards on P_i. Then we leave P_i, and if we hit a P_j (which may equal P_i), then we proceed backwards on P_j, etc. We shall assume that the part of P outside $P_1\cup P_2\cup\ldots\cup P_{k-1}$ consists of a finite collection of pairwise disjoint SA-s, whereas we allow overlap of the arcs of P in $P_1\cup P_2\cup\ldots\cup P_{k-1}$. For each $i=1,2,\ldots,k-1$, let a_i be the supremum of the elements on P_i to which there is an augmenting arc from an element a in A but not in $P_1\cup P_2\cup\ldots\cup P_{k-1}$. Now $M\setminus\{a_1,a_2,\ldots,a_{k-1}\}$ has an SA from A to B. Following this backwards from B, and using the definition of the a_i, we conclude that M has an augmenting arc from an element a in A to an element b in B such that none of a,b are ends of the arcs $P_1,P_2,\ldots,P_{k-1}$. Now the symmetric difference between this arc and $P_1\cup P_2\cup\ldots\cup P_{k-1}$ consists of a collection of $SCC-s$ and the desired k $SA-s$ from A to B. ∎

Lemma 2.4. *Let k be a natural number, and A a subset with k elements of an arcwise connected metric space M. Let x be an element in $M\setminus A$ such that M is locally 2-connected around x. (That is, every neighborhood around x contains a connected, locally 2-connected neighborhood around x.) Suppose furthermore that, for any set Q of $k-1$ elements in $M\setminus\{x\}$, $M-Q$ has an SA from A to x. Then M has k SA-s from A to x which are pairwise disjoint except that they all contain x.*

Proof. Suppose that the open disc $D(x,1)$ is disjoint from A. Let U_1 be an open, connected, locally 2-connected neighborhood of x in $D(x,1)$. Let B_1 be a set of k elements in U_1. Now apply Theorem 2.3 with B_1 instead of B. We follow each of the resulting $SA-s$ from A in the complement of U_1 until we reach an element in the boundary of U_1 such that any extension of the SA intersects U_1. Let A_1 be the ends of these $SA-s$. We now consider $U_1 \cup A_1, A_1, D(x,1/2)$ instead of $M, A, D(x,1)$, respectively and we apply again Menger's theorem as above. Before we apply Menger's theorem we add, if necessary, small $SA-s$ from A_1 each of diameter less than $1/4$ so that the resulting space is arcwise connected. We repeat this argument for smaller and smaller neighborhoods around x. The union of the resulting $SA-s$ together with x form the desired $SA-s$. ∎

Proposition 2.1 shows that any connected, locally 2-connected metric space satisfies the assumption and hence also the conclusion of Lemma 2.4.

3. Locally flat, locally 2-connected, compact metric spaces

The classification of surfaces tells us the global structure of spaces with a certain local structure. The result of this section extends that in the sense that we now relax the local conditions considerably.

Following [6] we say that a metric space is *flat* if it contains none of the Kuratowski graphs K_5 or $K_{3,3}$. It is easy to see that, if a 3-connected metric space (and hence also a locally 2-connected space) contains K_5, then it also contains $K_{3,3}$. So, for such spaces we may omit K_5 in the definition of being flat. We also say that a metric space is *locally flat* if, for every element x in the space, and every neighborhood U around x, there is a flat neighborhood V of x contained in U. In this section we prove that every locally flat, connected, locally 2-connected, compact metric space is contained in a surface.

Theorem 3.1. *If M is a locally flat, connected, locally 2-connected, compact metric space, then M is homeomorphic to a subset of a surface.*

Proof. Suppose that C is an SCC in M. By Proposition 2.1 in the present paper and Lemma 2.2 in [6], the overlap graph $O(M,C)$ is connected. (For the definition of overlap graph, see [6].) If it is bipartite, then it has a unique bipartition.

We say that C is *contractible* if the overlap graph is bipartite and one of the parts together with C form a set which is homeomorphic to a subset of the plane inside (the image of) C. We may assume that this part of $O(M,C)$ is unique as otherwise M is a subset of the sphere. This part together with C is called *the interior of* C and is denoted $Int(C,M)$. Also we put $int(C,M) = Int(C,M) \setminus C$. Using the afore-mentioned homeomorphism we extend M such that $Int(C,M)$ becomes homeomorphic to a closed disc in the plane. We shall use the expression that *we have filled in the holes inside C*. We claim that this process can be repeated for only finitely many C until we transform M into a compact metric space which is locally homeomorphic to either a disc or a half-disc. That space is homeomorphic to a surface with holes, and clearly M is homeomorphic to a subset of that surface.

Before we prove the claim we comment on the structure of the above mentioned $Int(C,M)$. Clearly, it is 2-connected. Clearly, $int(C,M)$ is locally 2-connected (but not necessarily connected). $Int(C,M)$ need not be locally 2-connected, but it is an easy exercise to prove that it is locally connected. Proposition 3 in [3] now implies that each face of $Int(C,M)$ (considered as a subset of the Euclidean plane) is bounded by an SCC. This justifies the above terminology that "we have filled in the holes inside C". If $Int(C,M)$ is a neighborhood around an element x in M, then we may replace C by an SCC C' in $Int(C,M)$ such that $Int(C',M)$ is a neighborhood around x and has the additional useful property: Every face boundary in $Int(C',M)$ has at most one element in common with C'. For, if C'' is a face boundary that has at least two elements in common with C, then we may replace a unique SA in C with a unique SA in C'' so that the new C together with its interior is a neighborhood around x. We do this for each C'' which has at least two elements in common with C. In the arguments below we perform this modification of C without explicitely saying so.

We now prove the claim. Let x be any element of M. We shall prove that every neighborhood of x contains a contractible SCC C_x such that x is either on or inside C_x. Moreover, if x is on C_x, then C_x together with its interior contains an open neighborhood U_x of x. (If x is not on C_x, we define U_x as $int(C_x,M)$.) Before we prove the existence of these C_x we explain why this proves the above claim and hence Theorem 3.1. Let δ be a positive number such that every $K_{3,3}$ in M has diameter at least 2δ. (If δ did not exist, then M would contain a sequence of $K_{3,3}-s$ whose diameter tends to zero. A subsequence would be convergent, and M would not be locally flat at the limit.) For each x, we may assume that C_x has diameter $< \delta$. (Otherwise we use the proof of Proposition 2.2 to replace C_x by a new C_x of diameter $< \delta$. More precisely, we first consider a new small C_x in the disc obtained by filling in the holes inside the old C_x. By the rerouting procedure of Proposition 2.2 we modify this new C_x so that it is contained in $int(C_x,M)$ and has small diameter.) As M is compact, there is a finite number of open neighborhoods U_x that cover M. We fill in successively the holes of the corresponding $SCC{-}s$ C_x and end up with a surface with holes. Before we find the C_x we shall address the following question. If we fill in $Int(C_x,M)$, and C_y is one of the subsequent $SCC-s$, how can we be sure that $Int(C_y,M)$ still exists? So let S be one of $SCC{-}s$ in $Int(C_x,M)$ which is filled out by a disc. Suppose that S intersects $int(C_y,M)$. Let p,q be any two elements of $S \cap int(C_y,M)$. We claim that p,q are on the same face boundary of $Int(C_y,M)$. For this, let us add an edge e from p to q, that is,

e is an SA having only p, q in common with M. If p, q were not on the same face boundary of $Int(C_y, M)$, then $Int(C_y, M)$ contains an SCC separating p and q by Corollary 4.4 in [6]. By Lemma 4.2 in [6], $Int(C_y, M) \cup \{e\}$ contains a $K_{3,3}$. Clearly also $Int(C_x, M) \cup Int(C_y, M) \cup \{e\}$ contains a $K_{3,3}$. But the proof of Proposition 2.2 implies that then $Int(C_x, M) \cup Int(C_y, M)$ contains a $K_{3,3}$. (Here, the rerouting is easier than in the proof of Proposition 2.2. The rerouting takes place in $Int(C_x, M)$ only around S and the holes close to S. We may use Proposition 2.2 because $int(C_x, M)$ is locally 2-connected. Moreover, when we reroute around S, we may choose the route so that we do not hit C_x because S has at most one element in common with C_x.) This contradiction to the definition of δ proves the claim that p, q are on the same face boundary of $Int(C_y, M)$. As p, q are in $int(C_y, M)$, this face boundary is distinct from C_y. So $S \cap Int(C_y, M)$ is contained in a face boundary R of $Int(C_y)$. Hence $S = R$ (because R is an SCC having at most one element in common with C_y), and therefore we do not get inconsistencies when we fill in the holes of C_x and C_y.

We shall now prove that the $SCC\!-\!s$ C_x exist. Let x_0 be any element and assume that the open disc $D(x_0, 3)$ is flat. As in the proof of the main result, Theorem 4.3, in [6], we define a sequence of 3-connected graphs $G_1, G_2, \ldots$ in $D(x_0, 1)$ and sets $A_1, A_2, \ldots$ such that A_n is $1/n$-dense in $D(x_0, 1)$. (See [6] for the relevant definitions. In [6] we worked in a compact space. Here we work in the open set $D(x_0, 1)$ so we must be careful when we use the results from [6]. In [6] we assumed that A_n is also $1/n$-dense in each face boundary. We cannot assume that here because A_n is contained in $D(x_0, 1)$ while face boundaries may intersect $C(x_0, 1)$.) In order to visualize G_n, we also draw it as a graph G'_n inside a square in the Euclidean plane, with all edges being polygonal, as in the proof of [6]. The advantage of working with 3-connected graphs is that such a graph has a unique embedding in the plane (once the outer face boundary has been specified). We define face boundaries of G_n and faces of G_n in $D(x_0, 3)$ as in [6]. We define a face of G_n to be *big* if it has an SA from the face boundary to $M \setminus D(x_0, 3)$. Otherwise the face is *small*. The number of big faces may increase as n increases. However it does not increase forever.

(1) There exists a natural number n_0 such that, for n sufficiently large, each G_n has precisely n_0 big faces.

Proof of (1). For each big face in G_n, we select an SA from the face to $C(x_0, 3)$ having only its ends in common with $G_n \cup C(x_0, 3)$. In G_{n+1} we can use (parts of) the same $SA-s$. In case an additional big face is created we select a corresponding SA in it. For each of these $SA-s$ we consider the first element in which it hits $C(x_0,2)$. If that results in an infinite set, then an accumulation point and a small connected neighborbood around that shows that the arcs do not belong to distinct faces, a contradiction which proves (1). ∎

We may construct the graphs G_n such that x_0 is on each of them, using the method in [6]. Using the local 2-connectedness we can avoid that x_0 gets degree larger than 2. The same argument ensures that we can let one of the two facial cycles containing x_0 have arbitrarily small diameter. Repeating the proof of (1), we may assume that, in addition, the facial cycle of small diameter is the boundary of a small face. If the other facial cycle can also get arbitrarily small diameter then the union of the two facial cycles containing x_0 contains a cycle that can play the role of C_{x_0}. So assume that the other facial cycle, which we call C_n and which we also call the outer cycle is always the boundary of a big face which we call F_n. Let F denote the intersection $F_1 \cap F_2 \cap \ldots$. As F is the intersection of connected, compact sets, F is connected (in the topological sense) and compact. Also, F contains x_0.

It follows from (1) in [6] that if y is an element which is both in F and $D(x_0,1)$, then y will be in F also if we modify the sequence $G_1,G_2,\ldots$.

When n is so big that (1) (in the present paper) holds and also the facial cycle containing x_0 of small diameter is the boundary of a small face, then we put $H_1 = G_n$. We forget about $G_{n+1},G_{n+2},\ldots$ (but we remember F) and define instead a new sequence $H_1,H_2,\ldots$ and the corresponding plane drawings $H_1',H_2',\ldots$ as follows. Let δ_0 denote the distance from H_1 to $C(x_0,1)$. Suppose that we have already defined H_n and δ_{n-1}. Suppose that we have also defined elements $x_1,x_2,\ldots,x_{n-1}$ in F occurring in that order on the outer cycle and SA-s $P_1,P_2,\ldots,P_{n-1}$ in $\bar{D}(x_0,1)$ such that P_i has its first element close to x_0 and between x_{i+1} and x_i in the big facial cycle (the outer cycle) B_n of H_n containing x_0, and P_i has its last element in $C(x_0,1)$ and no other element in common with $H_n \cup C(x_0,1)$ and such that $P_1,P_2,\ldots,P_{n-1}$ are pairwise disjoint except that they may intersect in $C(x_0,1)$. Then we define P_n and H_{n+1} and δ_n as follows. We let δ_n be a positive number smaller than $4^{-n}\delta_0$ and also smaller than the distance from x_0 to $P_1 \cup P_2 \cup \ldots \cup P_{n-1}$. Let U_n be an open neighborhood of diameter $< \delta_n$ around x_0 such that $U_n \cap B_n$ is an SA (without ends). Let V_n be an open connected neighborhood in U_n around x_0. As F is connected (in the topological sense), $F \cap V_n$ contains an element x_n distinct from x_0. Applying Lemma 2.4 to V_n, we may enlarge H_n to a 2-connected graph containing x_n by adding an SA in V_n. Using the local 2-connectivity we can further extend this graph to a 3-connected graph H_{n+1} by adding a finite number of arcs close to B_n. The arcs $P_1,P_2,\ldots,P_{n-1}$ may get a little shorter this way. In the resulting graph we consider the facial cycle (which we also call B_n) containing $x_0, x_1, \dots, x_n$. We consider the segment of B_n of diameter $< \delta_n$ between x_0 and x_n. Suppose M has a $B_n \cup P_{n-1}$-component which is attached to this small segment and which also intersects $C(x_0, 1)$. Then we let P_n be an SA in this component from B_n to $C(x_0, 1)$.

Suppose first that we obtain an infinite sequence $P_1, P_2, \dots$. Following P_n from B_n, let y_n be the first element in $C(x_0, 1 - \delta_0/2)$. Let y_0 be an accumulation point of $y_1, y_2, \dots$. In a connected neighborhood around y_0 of diameter $< \delta_0/4$ there is an SA connecting P_i and P_j where $j > i$. Then H_i can be extended to a 3-connected graph in $D(x_0, 1)$ such that x_{i+1} and x_0 are not on the same face boundary, contradicting that x_{i+1} and x_0 are both on F and therefore not separated by an SCC in $D(x_0, 1)$. (We are here using the fact that P_i and P_j must be drawn in the outer face of the current graph.)

We may therefore assume that there is an n so that P_n does not exist. Hence none of the $B_n \cup P_{n-1}$-components of M which are attached to the segment of B_n from x_0 to x_{n-1} in U_n intersect $C(x_0, 1)$. We consider now the union of H_n and all those $B_n \cup P_{n-1}$-components. It is an easy exercise to prove that this is a compact, 2-connected, locally connected metric space. H_n' can be extended to an embedding in the Euclidean plane of this space, by the proof of the main result in [6]. Using this embedding we may extend H_n to a 3-connected graph in $D(x_0, 1)$ having a facial cycle which contains an SA Q which starts at x_0 and is contained in F. (Informally, we have now found one half, namely Q, of the boundary of a closed half-disc containing x_0.) We now call this graph H_1 and repeat the above reasoning except that we now insist that x_i is on the segment from x_{i-1} to x_0 not intersecting Q. In this way we extend H_1 to a 3-connected graph such that x_0 is on precisely two facial cycles one of which contains a segment of F in which x_0 is an inner point. By applying [6] to the other face, we conclude that M has a neighborhood around x_0 homeomorphic to a subset of a half-disc. This completes the proof of Theorem 3.1. ∎

4. Locally 2-connected, compact metric spaces that are not locally flat

In this section we prove that every locally 2-connected, compact metric space that is not locally flat contains an infinite complete graph. An element x in a metric space M is called an *f-element* if some neighborhood around x is flat. Otherwise, x is called an *nf-element*. A *strong nf-element* is an element x such that each neighborhood around x contains infinitely many nf-elements.

Lemma 4.1. *Let x_1,x_2,y_1,y_2 be elements in a connected, locally 2-connected metric space M containing $K_{3,3}$. Then M contains two disjoint $SA-s$ P_1,P_2 such that P_i has ends x_i,y_i for $i=1,2$.*

Proof. Let Q denote a $K_{3,3}$ in M. Let X_1,X_2,Y_1,Y_2 be pairwise disjoint $SA-s$ from x_1,x_2,y_1,y_2 to Q. Let x'_1,x'_2,y'_1,y'_2 be the ends of these $SA-s$ in Q. If each of x'_1,x'_2,y'_1,y'_2 is a vertex of Q, it is easy to find the desired $SA-s$. If x'_1, say, is not a vertex but contained in an edge pq of Q, then we may assume that both of p,q are in $\{x'_2,y'_1,y'_2\}$. For otherwise we consider the segment of pq from p to x'_1 and we may assume that it contains none of x'_2,y'_1,y'_2. (For otherwise, we focus on one of these.) We let U be an open connected neighborhood containing this segment such that U intersects only those edges of Q which are incident with p. We consider the maximal initial segments of X_1 and the edges from the neighbors of p which are disjoint from U. Using (the proof of) Lemma 2.4 the remaining part of X_1,Q are now replaced by four $SA-s$ which are pairwise disjoint except that they all start at x'_1. Hence x'_1 also plays the role of p. So we may assume that two or three of x'_1,x'_2,y'_1,y'_2 are vertices of Q, and that the remaining element (or elements) is on an edge of Q connecting two of x'_1,x'_2,y'_1,y'_2. We may assume that either x'_2,y'_1,y'_2 are vertices of Q, $y'_1y'_2$ is an edge of Q, and x'_1 is on that edge, or else y'_1,x'_1,y'_2,x'_2 are on the edge $y'_1x'_2$ in that order. As the two cases are similar we consider only the latter. We consider a small connected neighborhood around y'_1 and add in it an SA from the edge $y'_1x'_2$ to one of the other two edges incident with y'_1 in Q. Similarly at x'_2. Now it is easy to find the two desired $SA-s$. ∎

Lemma 4.2. *Let k be a natural number, let A be a subset with k elements of a locally 2-connected metric space M. Let x be a strong nf-element in $M\setminus A$. Then M has k $SA-s$ from A to x which are pairwise disjoint except that they all contain x. Also, each of them contains a sequence of nf-elements converging to x.*

Proof. We prove the lemma by induction on k. Let us first assume that $k=2$. (The case $k=1$ is easier.) By Lemma 2.4, M has two $SA-s$ P_1,P_2 from A to x which are disjoint except that they both contain x. In any connected neighborhood U around x, there are, by Lemma 2.4, three $SA-s$ from $P_1\cup P_2$ to an nf-element y which are pairwise disjoint except that they all contain y. Two of these start at the same P_i and that P_i can then be modified so that it contains y. We perform this modification infinitely often for smaller and smaller neighborhoods U. If both of the modified P_1,P_2 contain infinitely many of the added nf-elements, we have finished. So assume that in some neighborhood around x there are no nf-elements on P_2. Using that M is locally 2-connected, we consider a sequence of smaller and smaller open connected neighborhoods around x and use these to find a sequence of disjoint paths $Q_1, Q_2, \ldots$ such that Q_i connects an element y_i in $P_1 - x$ with an an element z_i in $P_2 - x$. Consider now Q_i, Q_{i+1} such that P_1 has an nf-element y between y_i and y_{i+1}. We ignore the segment of P_2 between z_i and z_{i+1} and we let V be an open connected subset containing Q_i, Q_{i+1} and the segment of P_1 between y_i and y_{i+1} such that every element of V is close to one of those three arcs. We now follow P_1, P_2 from x and from A until these four arcs hit V. Then we use Lemma 4.1 to reroute P_1, P_2 inside V such that the first segment of P_i is followed by the last segment of P_{3-i}, $i = 1, 2$. We perform this modification infinitely often, closer and closer to x and obtain thereby the two desired arcs.

Assume now that $k > 2$. Let u be an element of A. We apply the induction hypothesis to $A \setminus \{u\}$. Then we let R be an SA from u to one of the $k-1$ arcs, say P. Let V be an open connected set containing R and $P - x$ such that V is disjoint from the other $k-2$ arcs from x. We shall modify R and $P - x$ inside V such that we obtain two arcs from x which, together with the $k-2$ other arcs form the desired k arcs. First let u_1 be an nf-element on P close to x. Then let V_1 be an open connected subset of V containing R and the segment of P from u_1 to A. We apply Lemma 2.4 to find two $SA-s$ in V_1 from u_1 to the two elements of $A \cap V_1$. We follow P from x until we hit the two arcs from u_1 and ignore the rest of P. Then we select an nf-element u_2 on P close to x, and we repeat the above argument with u_2 instead of u_1. We repeat this construction infinitely often. This results in two arcs from x to A which are disjoint except for x. It is possible that one of these does not contain an infinite sequence of nf-elements converging to x. But then we repeat the last argument from the case $k = 2$. In that case we used local connectedness around x to find the arcs $Q_1, Q_2, \ldots$. In the present case, they already exist in the configuration we have found. ∎

Theorem 4.3. *Let k be a natural number, and let A be a set consisting of k strong nf-elements in a connected, locally 2-connected metric space M. Then M contains a complete graph K_k with vertex set A.*

Proof. The proof is by induction on k. For $k \leq 2$, the theorem is trivial, so assume that $k \geq 3$. Let a be an element of A. By the induction hypothesis $M \setminus \{a\}$ contains a complete graph K_{k-1} with vertex set $A \setminus \{a\}$. By Lemma 4.2, M has k $SA-s$ from a to K_{k-1} which are pairwise disjoint except that they all contain a. Also each of them contains a sequence of nf-elements converging to a. Those which end in A will be edges in the K_k. Those which end on an edge of the K_{k-1} will be modified into edges in the K_k as follows. Consider an arc P which ends at an element u on an edge pq of the K_{k-1}

such that the segment of pq from p to u contains no end of an arc from a. If p is not already an end of an edge from a, we consider an open connected set V containing the segment of pq from p to u (except p) such that V is disjoint from all other edges of the K_{k-1} and all other arcs from a. Using Lemma 2.4 we modify the part of the edge pq and the part of P inside V so that the new P hits the K_{k-1} at p.

On the other hand, if p is already an end of an edge from a, then we consider a small open connected neighborhood U around p and we add in U simple arcs avoiding p and connecting the edges incident with p such that we obtain an arc from pq to an edge pv on which there is no end (distinct from p) of an arc from a. We shall modify P so that it ends at v. To do this we consider an arc P' from a to v consisting of P, a segment of pq, some arcs in U, and a segment of pv. This arc may intersect many edges of the K_{k-1}. For each intersection we use Lemma 4.1 to successively modify P and part of an edge of the K_{k-1} to get rid of the intersection. We explain the first step in this series of modifications. Let a' be an nf-element on P, and let p' be the last element of P' on the edge qp. Let V be an open connected set containing the segment of P' from a' to p' and intersecting no edge except pq. Let x_1, y_1, x_2, y_2 be the first elements that P' (followed from a and v, respectively) and the edge pq (followed from p and q, respectively) have in common with the boundary of V such that any extension of these arcs have infinitely elements in common with V. Now apply the proof of Lemma 4.1 to the union of V and small arcs starting at x_1, y_1, x_2, y_2, respectively. ∎

Theorem 4.4. *Let k be a natural number, and let A, B be disjoint subsets, each with k elements in a connected, locally 2-connected metric space M. Let G be a complete graph K_{3k} in M disjoint from A and B. Then M contains a collection of $2k$ pairwise disjoint SA-s from $A \cup B$ to $V(G)$ which hit and leave only those edges of G which are incident with two ends of the SA-s. In particular, if $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$, then M has a collection of pairwise disjoint SA-s with ends $a_i, b_i, i = 1, 2, \dots, k$.*

Proof. First we apply Theorem 2.3 with $A \cup B$ instead of A and a set of $2k$ vertices on $V(G)$ instead of B. Then we repeat the proof of Theorem 4.3. (The present proof is easier. We need not use Lemmas 2.4, 4.1 as we do not care about intersections between the SA-s which we find and those edges of G incident with two ends of the SA-s.) The last remark follows easily using the k vertices of G which are not ends of the SA-s. ∎

Theorem 4.5. *Let x_0 be an element in a locally 2-connected metric space M. Let $P_1, P_2, \dots$ be an infinite sequence of SA-s which are pairwise disjoint except that they all start at x_0. For each unordered pair i, j of distinct* *natural numbers, let $Q_{i,j}$ be an SA from P_i to P_j intersecting no other P_k. Assume that all $Q_{i,j}$ are pairwise disjoint. Assume furthermore that, for each positive real number ϵ, only finitely many of the SA-s $Q_{i,j}$ and P_k have diameter $>\epsilon$. Let x_i denote the end of P_i distinct from x_0, $i=1,2,\ldots$. Then M contains an infinite complete graph whose vertex set is contained in $\{x_1,x_2,\ldots\}$.*

Proof. Let $y_{i,j}$ be an element on $Q_{i,j}\setminus(P_i\cup P_j)$. Let $U_{i,j}$ be an open connected neighborhood around $y_{i,j}$ such that $U_{i,j}$ does not intersect any of the above mentioned arcs except $Q_{i,j}$ We first focus on the SA P_1. Let us assume that we meet $Q_{1,2},Q_{1,3},\ldots$ in that order when we traverse P_1 towards x_0. Suppose that we have already found n SA-s, say $R_1,R_2,\ldots,R_n$, from x_1 to $\{y_{1,2},y_{1,3},\ldots,y_{1,n}\}$ such that these SA-s are pairwise disjoint except that they all contain x_1. We follow P_1 from x_0 untill we hit some R_i. All the other R_j, $1\le j\le n, j\ne i$ will never be modified. We modify the union of R_i and $Q_{1,n+1}$ and the arc in P_1 connecting them in a small open set around these arcs such that we obtain SA-s $R_1,R_2,\ldots,R_{n+1}$ instead of $R_1,R_2,\ldots,R_n$. We repeat this process so that we get SA-s from x_1 to $\{y_{1,2},y_{1,3}\ldots\}$ (except possibly to one of these elements). We repeat this successively for $P_2,P_3,\ldots$. It is easy to glue the resulting systems of SA-s together to an infinite complete graph. ∎

Theorem 4.6. *Let M be a connected, locally 2-connected, compact metric space which is not locally flat. Then M contains an infinite complete graph.*

Proof. As M is not locally flat, it contains at least one nf-element.

We consider first the case where M contains an nf-element x_0 and a neighborhood U_0 around x_0 which contains no nf-element, except x_0. Let G_0 be a $K_{3,3}$ in U_0. As M is locally 2-connected, we may assume that G_0 does not contain x_0. (Otherwise, we modify G_0 in a small 2-connected neighborhood around x_0.) Let U_1 be a neighborhood around x_0 disjoint from G_0. Let G_1 be a $K_{3,3}$ in U_1 not containing x_0. Also, let P_1 be an SA from G_1 to G_0. We continue defining disjoint copies $G_2,G_3,\ldots$ of $K_{3,3}$ and $SA-s$ $P_2,P_3,\ldots$ closer and closer to x_0 such that P_k is an SA from G_k to $G_0\cup G_1\cup\ldots\cup G_{k-1}\cup P_1\cup P_2\cup\ldots\cup P_{k-1}$. This results in an infinite connected graph which we call H. As M is locally 2-connected we may assume that we never create a vertex of degree 4 or more in H.

For every element x in H we let V_x be an open 2-connected neighborhood around x. We first choose V_x for all the vertices of degree 3 in G_0, such that all these neighborhoods are pairwise disjoint, and then for all other elements in G_0, then for all vertices of G_1 etc., in such a way that V_x is disjoint from V_y unless x and y are on the same edge of H. (As the graphs $G_0,G_1,\ldots$

converge towards x_0, there are only finitely many graphs to worry about at each stage.)

In Theorem 3.1 we proved the following: For any element x of M, every neighborhood of x contains a contractible SCC C_x such that x is either on or inside C_x. Moreover, if x is on C_x, then C_x together with its interior contains an open neighborhood of x. All we used in that part of the proof is that M is compact, locally 2-connected, and locally flat around x. Therefore, the same argument applies to every element x of H. So we may assume that the above C_x is contained in V_x. Now $Int(C_x)$ contains an open set containing x. As every edge of H from p to q, say, is compact, it has a finite covering by these sets. In this finite covering there exist $y_1, y_2, \ldots, y_m$ such that $int(C_{y_1})$ contains p, $int(C_{y_m})$ contains q, and any two consequtive sets in the sequence $int(C_{y_1}), int(C_{y_2}), \ldots$ intersect, and any two nonconsecutive sets are disjoint. (If p, say, is on C_{y_1}, then we modify H such that p is moved to $int(C_{y_1})$, as claimed. If two nonconsecutive sets intersect, then we modify the edge pq in the union of these sets. In this way we modify the edges of H one by one.)

Consider now two consecutive sets, say $int(C_{y_1}), int(C_{y_2})$. They are both homeomorphic to subsets of discs in the plane, but they may have a complicated intersection. We concentrate on $Int(C_{y_1})$. We think of C_{y_1} as a circle in the plane with center p and radius 2. H has three arcs from p, and they hit the three neighboring discs at points of distance (in the plane) at most 1, say. $C_{y_2} \cap Int(C_{y_1})$ consists of at most countably many arcs. Only finitely many arcs intersect the circle with center p and radius 1. Using this observation we can use the method of Proposition 2.2 to modify (shrink) C_{y_1} and C_{y_2} inside $Int(C_{y_1})$ so that the new C_{y_1} and C_{y_2} have an SA in common. We do this for any two consecutive sets of the form $Int(C_{y_i})$.

Now we fill in the holes in each $Int(C_{y_i})$ as in the proof of Theorem 3.1. In Theorem 3.1, this results in a surface with holes. This is not the case in the present proof as the space we consider is not compact. If we add x_0 to it, we obtain a compact space which is locally homeomorphic to a disc or half-disc at every point except x_0. Using the classification theorem, it is easy to see that this space (that is, the union of all the $Int(C_{y_i})$ with the holes filled in) contains the space obtained from a closed disc in the plane by adding infinitely many handles or crosscaps converging to a point on the boundary. It is also an easy excecise to draw in it an increasing sequence of finite complete graphs whose union is an infinite complete graph. At each stage we modify the new edges by the method of Proposition 2.2 such that they are in M.

It only remains to discuss the case where each nf-element in M is a strong nf-element. Let x_0 be any nf-element. We shall use Theorems 4.3, 4.4 to show that we can find SA-s satisfying the condition of Theorem 4.5 which will complete the proof. Suppose U_n is an open connected neighborhood around x_0 of diameter at most $1/n$. Suppose also that we have already defined SA-s $P_{1,n}, P_{2,n}, \ldots, P_{n,n}$ each starting outside U_n and terminating inside U_n and such that there are pairwise disjoint arcs outside U_n joining any two of them before they hit U_n. We shall then define U_{n+1} and SA-s $P_{1,n+1}, P_{2,n+2}, \ldots, P_{n+1,n+1}$. As x_0 is a strong nf-element, U_n contains infinitely many nf-elements. By Theorem 4.4, U_n contains a complete graph K_{5n}. We may assume that it does cot contain x_0, and we let U_{n+1} be an open, connected neighborhood in U_n around x_0 and disjoint from K_{5n}. We also select a set B of $n+1$ elements in U_{n+1}. We let A be the last elements of $P_{1,n}, P_{2,n}, \ldots, P_{n,n}$ such that the initial arcs in $P_{1,n}, P_{2,n}, \ldots, P_{n,n}$ before A are outside U_n. Now we apply Theorem 4.4 to obtain $P_{1,n+1}, P_{2,n+2}, \ldots, P_{n+1,n+1}$. Continuing like this, we obtain the SA-s satisfying the condition of Theorem 4.5. ∎

Combining Proposition 2.2 and Theorems 3.1 and 4.6 we obtain the following classification result.

Theorem 4.7. *Let M be a connected, locally 2-connected, compact metric space. Then either M contains an infinite complete graph, or else M is homeomorphic to a surface S with holes. Moreover, M and S contain the same finite graphs.*

5. Applications of the classification theorem

Theorem 5.1. *Let G be a graph, let H be a subgraph of G and let M be a locally 2-connected, compact metric space containing H. Then M also contains G, if and only if there exists no surface S which contains H but not G.*

Proof. If M contains an infinite complete graph, then clearly M contains G. Now the surface S in Theorem 4.7 can play the role of S in Theorem 5.1. ∎

Theorem 5.1 has several remarkable consequences when we combine it with the theory of graph embeddings [2]. For example, if we embed the complete graph K_k on the surface with smallest Euler genus (which is the crosscap number or twice the genus), then that surface does not contain K_{k+1}, except when $k=5$. Thus we get:

Corollary 5.2. *Let M be a locally 2-connected, compact metric space. If M contains K_5, then M contains K_6.*

We also get:

Corollary 5.3. *Let M be a locally 2-connected, compact metric space, and let k be a natural number. If M contains the complete graph K_k minus an edge, Then M also contains K_k except possibly if $k=2 \pmod 3$.*

Corollary 5.4. *Let G be a (finite) graph. Then G is contained in every locally 2-connected, compact metric space if and only if G is planar.*

The *k-path problem* for graphs is the problem of deciding whether a graph with $2k$ prescribed distinct vertices $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$ has k pairwise disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i connects x_i, y_i for $i = 1, 2, \ldots, k$. This problem is difficult, and its solution is one of the highlights of the Robertson-Seymour theory [4]. This problem has a natural generalization to metric spaces, which we may call the *k-arc problem*. For surfaces with finitely many holes, the problem is easy. If x_1, say, is not on the boundary of a hole, or if x_1, y_1 are on the boundary of two distinct holes, then we just connect them by any arc and regard this arc (possibly together with one or two of the holes) to be a new hole. In this way we have reduced the problem to a $(k-1)$-arc problem. If any pair of prescribed elements are on the boundary of the same hole, then it is easy to decide, by a fast algorithm, if the desired arcs exist. If $k=2$, then the arcs exist unless all four prescribed elements are on the boundary of the same hole of the sphere. If $k=3$ and the surface is not the sphere, then the arcs exist except possibly when all six prescribed elements are on the boundary of the same hole in the projective plane. The following reduces the k-arc problem for compact, locally 2-connected metric space to the surface case.

Theorem 5.5. *Let M be a connected, compact, locally 2-connected metric space, and let k be a natural number. Let $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$ be distinct elements. Then M has k pairwise disjoint $SA-s$ $P_1, P_2, \ldots, P_k$ such that P_i connects x_i, y_i $i = 1, 2, \ldots, k$ unless M is homeomorphic to a subset of a surface with finitely may holes in which $P_1, P_2, \ldots, P_k$ do not exist.*

Proof. If M contains an infinite complete graph, then the arcs $P_1, P_2, \ldots, P_k$ exist by Theorem 4.4. Otherwise let S be as in Theorem 4.7. Now we repeat the proof of Proposition 2.2, except that we do no fill in those finitely many holes whose boundaries intersect $\{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$. ∎

By similar arguments we get

Theorem 5.6. *Let M be a compact, connected, locally 2-connected metric space, and let A be a finite subset of M. Then M contains an SCC containing all elements of A in any prescribed order unless M is homeomorphic to a subset of a surface with finitely many holes in which the SCC does not exist.*

The simplest case of Theorem 5.6 is where A has four elements. In that case the SCC exists unless M is a subset of the sphere with one hole, and the four elements of A are all on the boundary of that hole in a wrong order.

References

[1] J. C. Hocking and G. S. Young: *Topology*, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1961.

[2] B. Mohar and C. Thomassen: *Graphs on Surfaces*, Johns Hopkins University Press, Baltimore, 2001.

[3] R. B. Richter and C. Thomassen: 3-connected planar spaces uniquely embed in the sphere, *Trans. Amer. Math. Soc.* **99** (1992), 116–130.

[4] N. Robertson and P. D. Seymour: Graph minors XIII. The disjoint paths problem, *J. Combinatorial Theory, Ser. B* **354** (2002), 4585–4595.

[5] C. Thomassen: Jordan–Schönflies theorem and the classification of surfaces, *American Math. Monthly* **99** (1992), 116–130.

[6] C. Thomassen: The locally connected compact metric spaces embeddable in the plane, *Combinatorica* **24(4)** (2004), 699–718.

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